Galois cohomology seminar
Week 3 - Cup products

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1 Dimension shifting

Let G be a group, and let A be a G-module. Recall the definition from last week,

$$M^G(A) = \mathrm{CoInd}^G(A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$$

and recall that we had an injective map $A \to M^G(A)$. We define A^* to be the cokernel of this map, making the following short exact sequence of G-modules.

$$0 \to A \to M^G(A) \to A^* \to 0$$

The next lemma won't be used for a while, but now is a good time to state it.

Lemma 1.1. *The short exact sequence above is split exact.*

Proof. The map $A \to M^G(A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ is given by $a \mapsto (1 \mapsto a)$. Then

$$M^G(A) \to A \qquad \phi \mapsto \phi(1)$$

is a section which is a homomorphism of G-modules, so the sequence is split. □

Corollary 1.2. *Let A, B be G-modules.*

$$0 \to A \otimes B \to M^G(A) \otimes B \to A^* \otimes B \to 0$$

is also split exact.

Proof. This follows from the general fact that additive functors (such as $- \otimes B$) preserve split exact sequences. □

Remark 1.1. The long exact sequence associated to $0 \to A \to M^G(A) \to A^* \to 0$ is very useful, because the cohomology groups $H^i(G, M^G(A))$ vanish for $i \geq 1$.

$$0 \to H^0(G,A) \to H^0(G, M^G(A)) \to H^0(G, A^*) \to H^1(G,A) \to 0 \to H^1(G, A^*) \to H^2(G,A) \to 0 \to \cdots$$

Just looking at this long exact sequence proves the following lemma.

Lemma 1.3 (Dimension shifting)**.** *Let A be a G-module. For $i = 0$, the connecting homomorphism $H^0(G, A^*) \to H^1(G, A)$ is surjective, and for $i \geq 1$ the connecting homomorphisms give isomorphisms*

$$H^i(G, A^*) \cong H^{i+1}(G, A)$$

2 Cup products

2.1 Defining the cup product via cochains

Let G be a group and let A, B be G-modules. Recall the definition of i-cochains, just functions (with no restrictions) from G^i to A.

$$C^i = C^i(G, A) = \{f : G^i \to A\}$$

Recall also that we have boundary maps $d^i : C^i \to C^{i+1}$, and that by definition,

$$Z^i = \ker d^i \qquad B^i = \mathrm{im}\, d^{i-1} \qquad H^i(G,A) = Z^i / B^i$$

Definition 2.1. Let A, B be G-modules. We give $A \otimes_{\mathbb{Z}} B$ a G-module structure via

$$g \cdot (a \otimes b) = ga \otimes gb$$

Definition 2.2. The **preliminary cup product** is the map

$$C^i(G, A) \otimes_{\mathbb{Z}} C^j(G, B) \xrightarrow{\wedge} C^{i+j}(G, A \otimes_{\mathbb{Z}} B)$$

defined for $f \in C^i(G, A), f' \in C^j(G, B)$ by

$$(f \wedge f')(g_1, g_2, \ldots, g_{i+j}) = f(g_1, \ldots, g_i) \otimes g_1 g_2 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j})$$

Note that this is in fact a well defined map on the tensor product; that is, it respects addition in either component and moving integer multiplication "through the tensor."

Remark 2.1. Henceforth, we will use heavily abuse notation by using d to refer to all of the following:

1. The boundary maps d_A^i for the chain complex $C^i(G, A)$.
2. The boundary maps d_B^j for the chain complex $C^j(G, B)$.
3. The boundary maps $d_{A\otimes B}^{i+j}$ for the chain complex $C^k(G, A \otimes B)$.

The indices don't really affect any of the calculations, and they just muddle up the notation a lot.

Lemma 2.1. *Let $f \in C^i(G, A)$ and $f' \in C^j(G, B)$. Then*

$$d(f \wedge f') = df \wedge f' + (-1)^i f \wedge df'$$

Including indices, we get a more precise, but ugly version of the formula.

$$d_{A\otimes B}^{i+j}(f \wedge f') = \left(d_A^i f\right) \wedge f' + (-1)^i f \wedge \left(d_B^j f'\right)$$

Proof. Tedious calculation, relegated to the appendix in 3.1. □

Corollary 2.2. *Let $\wedge$ be the product defined above.*

1. *The product of two cocycles is a cocycle.*
2. *The product of two coboundaries is a coboundary.*
3. *The product of a coboundary with a cocycle is a coboundary.*

Proof. In this proof, we omit indices for the most part, since they don't really matter. (1) If $f \in Z^i(G, A) = \ker d^i$ and $f' \in Z^j(G, B) = \ker d^j$, then

$$d(f \wedge f') = df \wedge f' + (-1)^i f \wedge df' = 0 + 0 = 0$$

Thus $f \wedge f'$ is a cocycle.

(2) Let $f = d\alpha$ and $f' = d\alpha'$ be two coboundaries. Then

$$d(\alpha \wedge d\alpha') = d\alpha \wedge d\alpha + (-1)^n \alpha \wedge d^2\alpha' = d\alpha \wedge d\alpha' = f \wedge f'$$

Thus $f \wedge f$ is a coboundary.

(3) Suppose $f \in \ker d^i$ is a cocycle and $f' \in \operatorname{im} d^{j-1}$ is a coboundary, $f' = d\alpha'$. Then

$$d(f \wedge \alpha') = df \wedge \alpha' + (-1)^i f \wedge d\alpha' = 0 + (-1)^i f \wedge f'$$

For completeness, we include a version of the above formula with indices.

$$d^{i+j-1}(f \wedge \alpha') = d^i f \wedge \alpha' + (-1)^i f \wedge d^{j-1}\alpha' = 0 + (-1)^i f \wedge f'$$

Thus $f \wedge f' = \pm d(f \wedge \alpha')$, which is to say, $f \wedge f'$ is a coboundary. □

Definition 2.3. The **cup product** is the map

$$H^i(G, A) \otimes_{\mathbb{Z}} H^j(G, B) \xrightarrow{\cup} H^{i+j}(G, A \otimes_{\mathbb{Z}} B)$$

defined in terms of the preliminary cup product map by

$$\overline{f} \cup \overline{f'} := \overline{f \wedge f'}$$

where $f \in Z^i(G, A)$ and $f' \in Z^j(G, B)$. The bars denote equivalence class $H^i(G, A)$, $H^j(G, B)$, and $H^{i+j}(G, A \otimes B)$ where appropriate.

Remark 2.2. By part (1) of the previous corollary, $f \wedge f'$ is a cocycle, so $\overline{f \wedge f'}$ makes sense as an element of $H^{i+j}(G, A \otimes B)$. To show this is well defined, we need to verify that the class $\overline{f \wedge f'}$ in $H^{i+j}(G, A \otimes B)$ is independent of choice of representatives $f \in Z^i, f' \in Z^j$.

Suppose we replace f, f' with alternate representatives $\widetilde{f} \in Z^i(G, A), \widetilde{f'} \in Z^j(G, B)$, then

$$\widetilde{f} = f + \phi \qquad \widetilde{f'} = f' + \phi'$$

where $\phi \in B^i(G, A), \phi' \in B^j(G, B)$. Then

$$\widetilde{f} \wedge \widetilde{f'} = (f + \phi) \wedge (f' + \phi') = f \wedge f' + f \wedge \phi' + \phi \wedge f' + \phi \wedge \phi'$$

Then taking equivalence classes in $H^{i+j}(G, A \otimes B)$, we get

$$\overline{\widetilde{f} \wedge \widetilde{f'}} = \overline{f \wedge f'} + \overline{f \wedge \phi'} + \overline{\phi \wedge f'} + \overline{\phi \wedge \phi'}$$

By part (3) of the corollary, $f \wedge \phi'$ and $\phi \wedge f'$ are coboundaries, so those terms vanish, and by part (2) of the corollary, the $\phi \wedge \phi'$ term vanishes. Thus $\cup$ is well defined.

Remark 2.3. There are alternative approaches to defining the cup product which are more in the flavor of homological algebra, but these take more machinery, such as tensor product of chain complexes. It can make some things easier to prove later.

2.2 Properties of cup product

Proposition 2.3 (Cup product in degree zero)**.** *In the case $i = j = 1$, the cup product is just the inclusion*

$$A^G \otimes_{\mathbb{Z}} B^G \hookrightarrow (A \otimes_{\mathbb{Z}} B)^G$$

Proof. There's nothing complicated to prove here, just some thinking to do. In terms of cochains, an element $f \in C^0(G, A)$ is a function $G^0 \to A$. By convention, G^0 is the trivial group, so $f \in C^0(G, A)$ is essentially a point in A. Lying in the kernel of $d^0 : C^0(G, A) \to C^1(G, A)$ means that

$$(d^0 f)(g) = gf - f = 0 \qquad \forall g \in G$$

which is to say, $gf = f$, which is to say, $f \in A^G$. Of course, we already knew this. But now thinking about the definition of cup product in terms of cochains, for $f, f' \in C^0(G, A)$,

$$(f \wedge f') = f \otimes f' \in A^G \otimes B^G \subset (A \otimes B)^G$$

$\square$

Proposition 2.4 (Naturality of cup product)**.** *The cup product is "natural" in A and B.*

More precisely, suppose $\phi : A \to A'$ is a morphism of G-modules and $\phi_* : H^i(G, A) \to H^i(G, A')$ is the induced map on homology. Note that $\phi \otimes 1 : A \otimes B \to A' \otimes B$ is a morphism of G-modules, and let $(\phi \otimes 1)_* H^i(G, A \otimes B) \to H^i(G, A' \otimes B)$ be the induced map on homology. Then for $\alpha \in H^i(G, A)$ and $\beta \in H^j(G, B)$, we have

$$\phi_*(\alpha) \cup \beta = (\phi \otimes 1)_*(\alpha \cup \beta)$$

Another way to state this is with the following commutative diagram.

$$\begin{array}{ccc} H^i(G, A) \otimes_{\mathbb{Z}} H^j(G, B) & \xrightarrow{\cup} & H^{i+j}(G, A \otimes_{\mathbb{Z}} B) \\ \big\downarrow{\scriptstyle H^i(\phi)\otimes \mathrm{Id}} & & \big\downarrow{\scriptstyle H^{i+j}(\phi \otimes \mathrm{Id})} \\ H^i(G, A') \otimes_{\mathbb{Z}} H^j(G, B) & \xrightarrow{\cup} & H^{i+j}(G, A' \otimes_{\mathbb{Z}} B) \end{array}$$

The analogous property holds for a morphism $\psi : B \to B'$.

Proposition 2.5 (Cup product "commutes" with connecting homomorphisms)**.** *If*

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

is a short exact sequence of G modules, and B is a G-module such that the sequence

$$0 \to A_1 \otimes B \to A_2 \otimes B \to A_3 \otimes B \to 0$$

is also exact, then for $\alpha \in H^i(G, A_3)$ and $\beta \in H^j(G, B)$,

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta \in H^{i+j+1}(G, A_1 \otimes B)$$

where the δ maps are connecting homomorphism coming from the long exact sequences. We can write this as a commutative diagram

$$\begin{array}{ccc} H^i(G, A_3) \otimes H^j(G, B) & \xrightarrow{\cup} & H^{i+j}(G, A_3 \otimes B) \\ \big\downarrow{\scriptstyle \delta \otimes 1} & & \big\downarrow{\scriptstyle \delta} \\ H^{i+1}(G, A_1) \otimes H^j(G, B) & \xrightarrow{\cup} & H^{i+j+1}(G, A_1 \otimes B) \end{array}$$

Proof. This is basically proved by reproving the snake lemma. You work through the whole process of lifing, etc. that the snake lemma uses to construct the connecting homomorphisms. Nothing too fancy, though not super easy to follow. ☐

Remark 2.4. There is an analogous property with tensoring applied on the other side of the exact sequence, though there is a slight issue of a sign, so the resulting equation becomes

$$\delta(\alpha \cup \beta) = (-1)^i \alpha \cup (\delta\beta)$$

where $\alpha \in H^i(G, A), \beta \in H^j(G, B_3)$.

Proposition 2.6 (Uniqueness). *The cup products we have defined are the unique family of maps satisfying the properties above (naturality, description in degree zero, and interaction with connecting homomorphisms).*

Proof. Suppose we have a product with these properties. We show by dimension shifting that the cup products in degree (i, j) determine cup products in degrees $(i + 1, j)$ and $(i, j + 1)$. Consider the short exact sequence

$$0 \to A \to M^G(A) \to A^* \to 0$$

and recall from earlier that it remains exact after tensoring with any B.

$$0 \to A \otimes B \to M^G(A) \otimes B \to A^* \otimes B \to 0$$

So by the property of coproduct, we get

$$\begin{array}{ccc} H^i(G, A^*) \otimes H^j(G, B) & \xrightarrow{\cup} & H^{i+j}(G, A^* \otimes B) \\ \downarrow{\scriptstyle \delta\otimes 1} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(G, A) \otimes H^j(G, B) & \xrightarrow{\cup} & H^{i+j+1}(G, A \otimes B) \end{array}$$

Recall that the connecting homomorphisms in the LES associated to $0 \to A \to M^G(A) \to A^* \to 0$ are isomorphisms for $i \geq 1$ and surjective for $i = 0$. Thus the cup product on the bottom is determined by the cup product on the top.

A similar argument using the other cup product property interacting with connecting homomorphisms shows that the cup products in degree (i, j) determine those in degree $(i, j + 1)$. ☐

Proposition 2.7 (Antisymmetry of cup product). *Let $\tau : A \otimes B \to B \otimes A$ be the canonical isomorphism ("twist map") $a \otimes b \mapsto b \otimes a$, and let $\tau_* : H^k(G, A \otimes B) \to H^k(G, B \otimes A)$ be the induced isomorphism on cohomology. For $\alpha \in H^i(G, A)$ and $\beta \in H^j(G, B)$,*

$$\tau_*(\alpha \cup \beta) = (-1)^{ij} \beta \cup \alpha$$

Since τ, τ_* are such natural/canonical isomorphisms, this is usually just written as

$$\alpha \cup \beta = (-1)^{ij} \beta \cup \alpha$$

Proof. Prove first in case $i = j = 0$, then use some dimension shifting to induct. $\square$

Proposition 2.8 (Associativity of cup product)**.** *The cup product is associative. More precisely, let C be another G-module, and let $\alpha \in H^i(G, A), \beta \in H^j(G, B), \gamma \in H^k(G, C)$. Then*

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma) \in H^{i+j+k}(G, A \otimes B \otimes C)$$

Really, these things don't quite live in the same homology group, but canonical isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ induced canonical isomorphisms between the homology groups that they live in.

Proof. This can be checked directly on the level of cochains, which is tedious. $\square$

Proposition 2.9 (Cup product and Res, Inf, Cor)**.** *Under suitable hypotheses and sufficient abuse of notation,*

$$\begin{aligned} \mathrm{Res}(\alpha \cup \beta) &= \mathrm{Res}(\alpha) \cup \mathrm{Res}(\beta) \\ \mathrm{Inf}(\alpha \cup \beta) &= \mathrm{Inf}(\alpha) \cup \mathrm{Inf}(\beta) \\ \mathrm{Cor}(\alpha) \cup \beta &= \mathrm{Cor}(\alpha \cup \mathrm{Res}(\beta)) \end{aligned}$$

2.3 The cohomology ring

Let G be a group and A a G-module. Then there are cup product maps

$$\cup : H^i(G, A) \times H^j(G, A) \to H^{i+j}(G, A \otimes_{\mathbb{Z}} A)$$

There is a morphism of G-modules

$$A \otimes_{\mathbb{Z}} A \to A \qquad a \otimes a' \mapsto a + a'$$

which induces a map on cohomology

$$H^i(G, A \otimes_{\mathbb{Z}} A) \to H^i(G, A)$$

We are just going to be sloppy and denote the composition of these by $\cup$.

$$\cup : H^i(G, A) \times H^j(G, A) \to H^{i+j}(G, A)$$

Thus we have a graded ring structure on

$$\bigoplus_{i=0}^{\infty} H^i(G, A)$$

In principle, the structure of this ring could differentiate between G-modules which have the same cohomology groups. Even if $H^i(G, A) \cong H^i(G, B)$ for all i, the cup products could be different, so we would conclude $A \ncong B$ as G-modules.

2.4 Application - cup product giving isomorphisms for cohomology of cyclic groups

I'm going to repeat a calculation I did before in more generality. Let $G = \mathbb{Z}/n\mathbb{Z}\langle\sigma\rangle$ be a finite cyclic group, and let A be any G-module (not necessarily trivial). Then we have the following projective (actually free) resolution of the trivial G-module $\mathbb{Z}$.

$$\cdots \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where

$$N = \sum_{i=1}^{n} \sigma^i$$

is the norm element of $\mathbb{Z}[G]$. We then apply (contravariant) $\mathrm{Hom}_{\mathbb{Z}[G]}(-, A)$ and drop the $\mathbb{Z}$ term to obtain the chain complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) & \longrightarrow & \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & A & \xrightarrow{\sigma-1} & A & \xrightarrow{N} & A & \xrightarrow{\sigma-1} & \cdots
\end{array}$$

where $\sigma - 1 : A \to A$ is the map $a \mapsto (\sigma - 1) \cdot a$, given by the $\mathbb{Z}[G]$-action on A. Thus

$$H^i(G, A) \cong \begin{cases} \ker(\sigma - 1) = A^G & i = 0 \\ \ker N/(\sigma - 1)A & i = 1, 3, \ldots \\ A^G/NA & i = 2, 4, \ldots \end{cases}$$

In the case where A is a trivial module, this simplifies as

$$H^i(G, A) \cong \begin{cases} A & i = 0 \\ n\text{-torsion subgroup of } A & i = 1, 3, \ldots \\ A/nA & i = 2, 4, \ldots \end{cases}$$

In particular,

$$H^2(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

We can describe the isomorphisms

$$H^2(G, A) \cong H^4(G, A) \cong \ldots \qquad H^1(G, A) \cong H^3(G, A) \cong \ldots$$

somewhat explicitly in terms of cup product with the generator of $H^2(G, \mathbb{Z})$.

Theorem 2.10 (CSAGC 3.4.11b). *Let $G \cong \mathbb{Z}/n\mathbb{Z}\langle\sigma\rangle$ be a finite cyclic group of order n. Fix a generator α of $H^2(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. The maps*

$$H^i(G, A) \to H^{i+2}(G, A) \qquad x \mapsto x \cup \alpha$$

are isomorphisms for $i \geq 1$.

3 Appendix 1 - a tedious calculation

Lemma 3.1. *Let $f \in C^i(G, A)$ and $f' \in C^j(G, B)$. Then*

$$d(f \wedge f') = df \wedge f' + (-1)^i f \wedge df'$$

Proof.

$$\begin{aligned}
\text{LHS} &= d(f \wedge f')(g_0, \ldots, g_{i+j}) \\
&= g_0 \cdot (f \wedge f')(g_1, \ldots, g_{i+j}) \\
&\qquad + \sum_{k=1}^{i+j} (-1)^k (f \wedge f')(g_0, \ldots, g_{k-1}g_k, \ldots, g_{i+j}) \\
&\qquad + (-1)^{i+j+1} (f \wedge f')(g_0, \ldots, g_{i+j-1}) \\
&= g_0 \cdot f(g_1, \ldots, g_i) \otimes g_1 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\qquad + \sum_{k=1}^{i} (-1)^k (f \wedge f')(g_0, \ldots, g_{k-1}g_k, \ldots, g_{i+j}) \\
&\qquad + \sum_{k=i+1}^{i+j} (-1)^k (f \wedge f')(g_0, \ldots, g_{k-1}g_k, \ldots, g_{i+j}) \\
&\qquad + (-1)^{i+j+1} f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} f(g_i, \ldots, g_{i+j-1}) \\
&= g_0 f(g_1, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\qquad + \sum_{k=1}^{i} (-1)^k f(g_0, \ldots, g_{k-1}g_k, \ldots, g_i) \otimes g_0 \cdots g_i f(g_{i+1}, \ldots, g_{i+j}) \\
&\qquad + \sum_{k=i+1}^{i+j} (-1)^k f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} f(g_i, \ldots, g_{k-1}g_k, \ldots, g_{i+j}) \\
&\qquad + (-1)^{i+j+1} f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} f(g_i, \ldots, g_{i+j-1})
\end{aligned}$$

$$
\begin{aligned}
\text{RHS} &= (df \wedge f' + (-1)^i f \wedge df')(g_0, \ldots, g_{i+j}) \\
&= (df \wedge f')(g_0, \ldots, g_{i+j}) + (-1)^i (f \wedge df')(g_0, \ldots, g_{i+j}) \\
&= df(g_0, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + (-1)^i f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} df'(g_i, \ldots, g_{i+j}) \\
&= g_0 f(g_1, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + \sum_{k=1}^{i} f(g_0, \ldots, g_{k-1}g_k, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + (-1)^{i+1} f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + (-1)^i f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} df'(g_i, \ldots, g_{i+j}) \\
&= g_0 f(g_1, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + \sum_{k=1}^{i} (-1)^k f(g_0, \ldots, g_{k-1}g_k, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + (-1)^{i+1} f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + (-1)^i f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} g_i f'(g_{i+1}, \ldots, g_{i+j}) + \\
&\quad + (-1)^i f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} \sum_{k=i+1}^{i+j} (-1)^{k-i} f'(g_i, \ldots, g_{k-1}g_k, \ldots, g_{i+j}) \\
&\quad + (-1)^i f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} (-1)^{j+1} f'(g_i, \ldots, g_{i+j-1}) \\
&= g_0 f(g_1, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + \sum_{k=1}^{i} (-1)^k f(g_0, \ldots, g_{k-1}g_k, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + (-1)^{i+1} f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + (-1)^i f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} g_i f'(g_{i+1}, \ldots, g_{i+j}) + \\
&\quad + \sum_{k=i+1}^{i+j} (-1)^k f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} f'(g_i, \ldots, g_{k-1}g_k, \ldots, g_{i+j}) \\
&\quad + (-1)^{i+j+1} f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} f'(g_i, \ldots, g_{i+j-1}) \\
&= g_0 f(g_1, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + \sum_{k=1}^{i} (-1)^k f(g_0, \ldots, g_{k-1}g_k, \ldots, g_i) \otimes g_0 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}) \\
&\quad + \sum_{k=i+1}^{i+j} (-1)^k f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} f'(g_i, \ldots, g_{k-1}g_k, \ldots, g_{i+j}) \\
&\quad + (-1)^{i+j+1} f(g_0, \ldots, g_{i-1}) \otimes g_0 \cdots g_{i-1} f'(g_i, \ldots, g_{i+j-1})
\end{aligned}
$$

□

4 Appendix 2 - explicit cocycle generators for $H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$

We know that if $G = \mathbb{Z}/n\mathbb{Z}$, then $H^2(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. We want to find a cocycle representing the generator of this cohomology group, that is, a function

$$f : G^2 \to \mathbb{Z}$$

satisfying the cocycle condition

$$f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0 \qquad \forall x, y, z \in G$$

which has order n in $H^2(G, \mathbb{Z})$. (The cocycle condition would usually have an extra factor of G-action on the $f(y, z)$ term, but it is trivial since $\mathbb{Z}$ is a trivial G-module.) We will switch to writing G additively, in which case this becomes

$$f(y, z) - f(x + y, z) + f(x, y + z) - f(x, y) = 0$$

The coboundaries are of the form

$$(d\phi)(x, y) = \phi(x) - \phi(x + y) + \phi(y)$$

where $\phi : G \to \mathbb{Z}$ is any function.

4.1 General remarks

We write the group $G = \mathbb{Z}/n\mathbb{Z}$ additively as integers $0, \dots, n - 1$ modulo n. The cocycle condition becomes

$$f(j, k) - f(i + j, k) + f(i, j + k) - f(i, j) = 0$$

for $i, j, k \in \mathbb{Z}/n\mathbb{Z}$. Taking $j = 0$ (which is like taking $y = 1$ in the original cocycle equation), we get

$$f(j, 0) = f(i, 0) \qquad \forall i, j$$

which implies

$$f(j, 0) = f(0, j) = f(i, 0) = f(0, i) \qquad \forall i, j$$

Additionally, setting $i = 0$ or $k = 0$ gives only equations which are redudant after having the above. Consequently, we can omit writing down any cocycle equations involving 0 in the future. If $i = k$ and $j = -i$, we get

$$f(-i, i) = f(i, -i)$$

If $i = j = k$, we get

$$f(2i, i) = f(i, 2i)$$

For coboundaries, we have the general calculation

$$(d\phi)(i, 0) = (d\phi)(j, 0) = \phi(0) \qquad \forall i, j$$

so all coboundaries satisfy the same general conditions from above, which are basically redundant for cocycles. Hence in the future, we can safely omit equations involving zero for coboundaries.

As an additional remark, the coboundary condition is perfectly symmetric with respect to the two variables, in the sense that $(d\phi)(i, j) = (d\phi)(j, i)$, since G is abelian. Therefore, one of these will always be a redundant equation, and we can avoid writing it down.

4.2 The case $n = 2$

In the case $n = 1$ there is nothing to say, since in that case $H^2(G, \mathbb{Z})$ is the trivial group. The next most simple case is $n = 2$, $G = \mathbb{Z}/2\mathbb{Z}$. Note that $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The cocycle condition is just 8 equations. However, we already know what happens when any of i, j, k is zero, namely

$$f(0,0) = f(1,0) = f(0,1)$$

The only remaining equation is when $i = j = k = 1$, which in this case ends up being redundant:

$$f(2,1) = f(1,2) \iff f(0,1) = f(1,0)$$

This covers all the cocycle conditions. That is, $f : G^2 \to \mathbb{Z}$ is a cocycle if and only if

$$f(0,0) = f(1,0) = f(0,1)$$

and $f(1,1)$ can be any integer. Now we seek necessary and sufficient conditions for a function $\phi : G \to \mathbb{Z}$ to be a coboundary. By preceeding general remarks, equations coming from $(d\phi)(i,j)$ with $i = 0$ or $j = 0$ are automatically satisfied by any cocycle, noting that $(d\phi)(0,0) = f(0,0)$ if f is a coboundary. So we just need to consider $(d\phi)(1,1)$.

$$\begin{aligned} (d\phi)(0,0) &= \phi(0) - \phi(0) + \phi(0) = \phi(0) \\ (d\phi)(1,1) &= \phi(1) - \phi(2) + \phi(1) = 2\phi(1) - \phi(0) \end{aligned}$$

Thus a cocycle $f : G^2 \to \mathbb{Z}$ is a coboundary if and only if

$$f(1,1) = 2a - f(0,0)$$

That is to say, a cocycle f is a coboundary if and only if $f(0,0) + f(1,1)$ is even. So f represents a nonzero class in $H^2(G, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ if and only $f(0,0) + f(1,1)$ is odd. So a representative for the generator (the only nonzero class) is given by the function

$$f(0,0) = f(1,0) = f(0,1) = 0 \qquad f(1,1) = 1$$

4.3 The case $n = 3$

Set $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. The cocycle condition for $f \in Z^2(G, \mathbb{Z})$ is now 27 total equations. However, we don't need to write down the ones involving zeros, since we already know they are equivalent to the condition

$$f(0,0) = f(1,0) = f(0,1) = f(2,0) = f(0,2)$$

The cases $i = j = k = 1, 2$ give

$$f(2,1) = f(1,2) \qquad f(4,2) = f(2,4)$$

which are the same equation modulo 3. We also covered the case $i = k, j = -i$ in general to obtain

$$f(-1,1) = f(1,-1) \qquad f(-2,2) = f(2,-2)$$

which, written modulo 3, are just both the equation $f(1,2) = f(2,1)$ again. The remaining equations to consider are

$$\begin{array}{ll} (i,j,k) & \\ (2,1,1) & 0 = f(1,1) - f(3,1) + f(2,2) - f(2,1) \\ (1,1,2) & 0 = f(1,2) - f(2,2) + f(1,3) - f(1,1) \\ (1,2,2) & 0 = f(2,2) - f(3,2) + f(1,4) - f(1,2) \\ (2,2,1) & 0 = f(2,1) - f(4,1) + f(2,3) - f(2,2) \end{array}$$

We reducing modulo 3, and do some substitutions, to get

$$\begin{aligned} 0 &= f(1,1) - f(0,0) + f(2,2) - f(1,2) \\ 0 &= f(1,2) - f(2,2) + f(0,0) - f(1,1) \\ 0 &= f(2,2) - f(0,0) + f(1,1) - f(1,2) \\ 0 &= f(1,2) - f(1,1) + f(0,0) - f(2,2) \end{aligned}$$

Shuffling some terms, we see that these are in fact all the same equation. Thus a function $f : G^2 \to \mathbb{Z}$ is a cocycle if and only if the following two equations hold.

$$f(1,2) = f(2,1) \qquad f(0,0) = f(0,1) = f(1,0) = f(2,0) = f(0,2) = f(1,1) - f(1,2) + f(2,2)$$

That is, $f(1,1), f(1,2), f(2,2) \in \mathbb{Z}$ are arbitrary, and the rest are determined by these.

Now we want to write down necessary and sufficient conditions for f to be a coboundary. Let $\phi : G \to \mathbb{Z}$ be any function. Recall that writing down $(d\phi)(0,i), (d\phi)(j,0)$ is redundant past writing down the equation for $i = j = 0$. The equation for $(d\phi)(2,1)$ is redundant as well, so we omit it.

$$\begin{aligned} (d\phi)(0,0) &= \phi(0) \\ (d\phi)(1,1) &= \phi(1) - \phi(2) + \phi(1) = 2\phi(1) - \phi(2) \\ (d\phi)(1,2) &= \phi(1) - \phi(0) + \phi(2) \\ (d\phi)(2,2) &= \phi(2) - \phi(1) + \phi(2) = 2\phi(2) - \phi(1) \end{aligned}$$

Let $(a,b,c) = (\phi(0), \phi(1), \phi(2))$. The previous equations then say that a cocycle $f : G^2 \to \mathbb{Z}$ is a coboundary if and only if there exist $a, b, c \in \mathbb{Z}$ such that

$$f(0,0) = a \qquad f(1,1) = 2b - c \qquad f(1,2) = b - a + c \qquad f(2,2) = 2c - b$$

We can immediately eliminate a, and solve each equation for b, to obtain

$$b = \frac{1}{2}\Big(f(1,1) + c\Big) = f(1,2) + f(0,0) - c = 2c - f(2,2)$$

This is really three equations, between each of the expressions involving c. After some manipulation, they become

$$\begin{aligned} 3c &= 2f(1,2) + 2f(0,0) - f(1,1) \\ 3c &= 2f(2,2) + f(1,1) \\ 3c &= f(1,2) + f(0,0) + f(2,2) \end{aligned}$$

We already know that $f(0,0) = f(1,1) - f(1,2) + f(2,2)$. Performing this substitution, we see that the three equations area all the same equation. Thus a cocycle f is a coboundary if and only if $2f(2,2) + f(1,1)$ is divisible by 3. Thus, a nonzero cocycle with order three is given by

$$f(0,0) = f(0,1) = f(1,0) = f(2,0) = f(0,2) = f(1,2) = f(2,1)$$
$$f(1,1) = 1 \qquad f(2,2) = 2$$